

A Game-theoretic Approach to Identify Key Nodes in Networks

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Supervisor

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Bird's-eye view

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Graph Theory
(social networks)

Bird's-eye view

Graph Theory
(social networks)

Problem:

how important is a node in a network?

Bird's-eye view

Graph Theory
(social networks)

Problem:

how important is a node in a network?

Answer: Centrality
Metrics

Bird's-eye view

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how important is a player in a game?

Bird's-eye view

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how important is a player in a game?

Answer: Shapley Value

Bird's-eye view

Graph Theory
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Shapley Value based centrality
metrics

```
graph TD; A[Graph Theory (social networks)] --> B[Problem: how important is a node in a network?]; B --> C[Answer: Centrality Metrics]; D[Game Theory] --> E[Problem: how important is a player in a game?]; E --> F[Answer: Shapley Value]; C --> G[Shapley Value based centrality metrics]; F --> G;
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Applications and future work

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Game Theory

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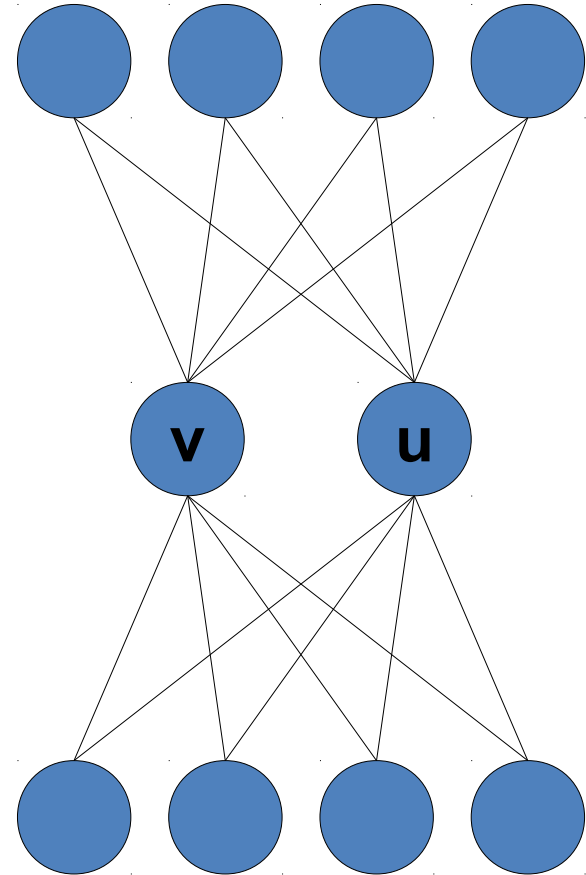
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Shapley Value based centrality
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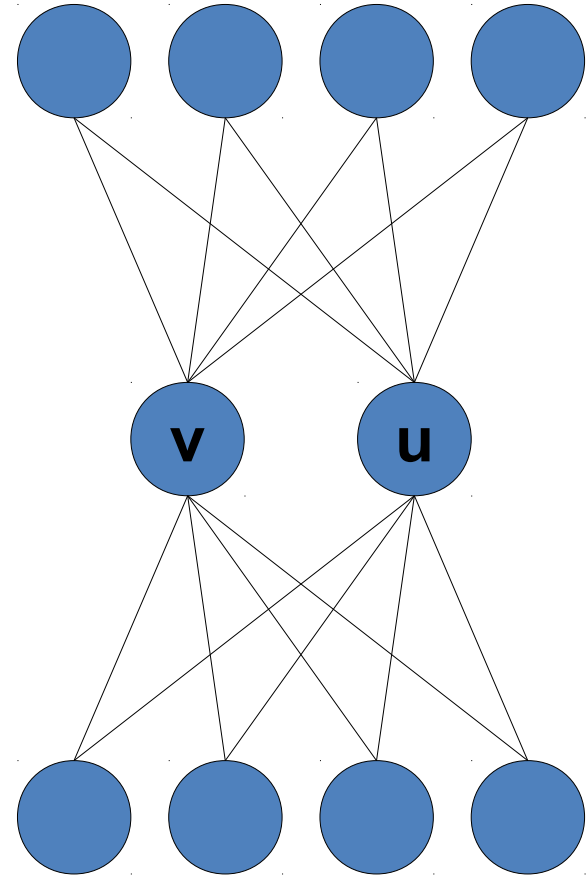
Applications and future work

Degree centrality



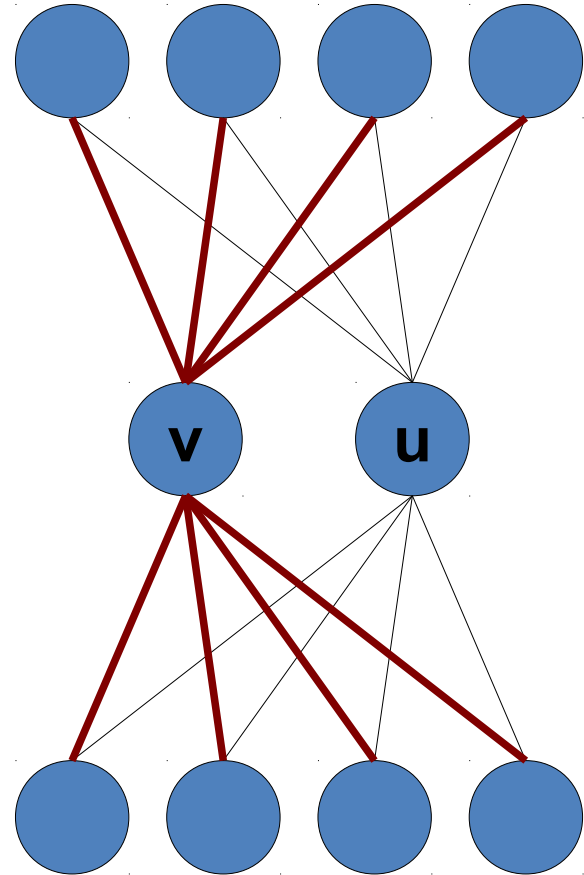
Degree centrality

- **Degree centrality** – how many edges a node **v** has



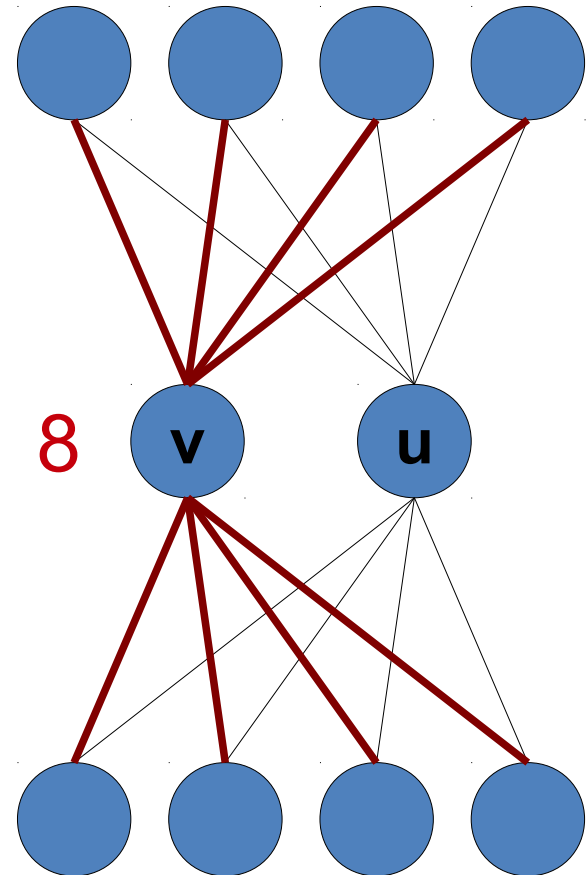
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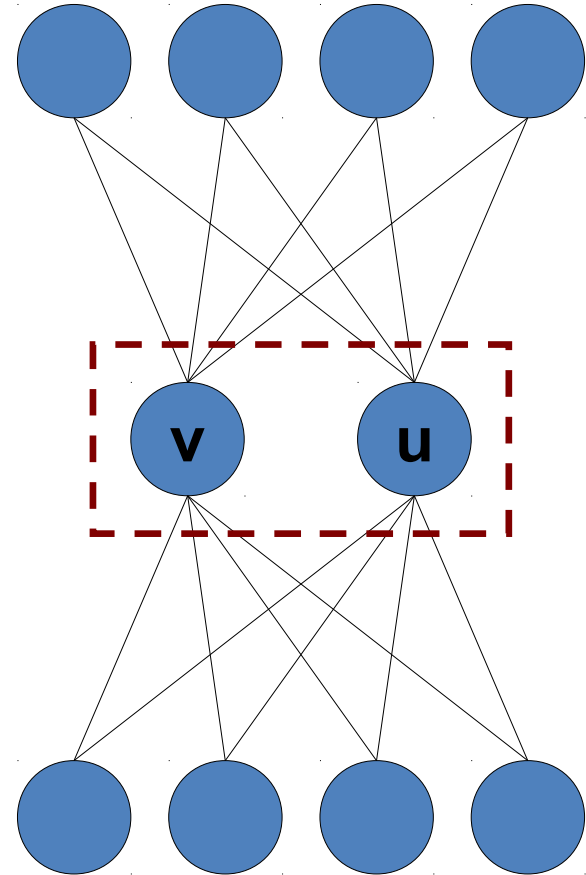
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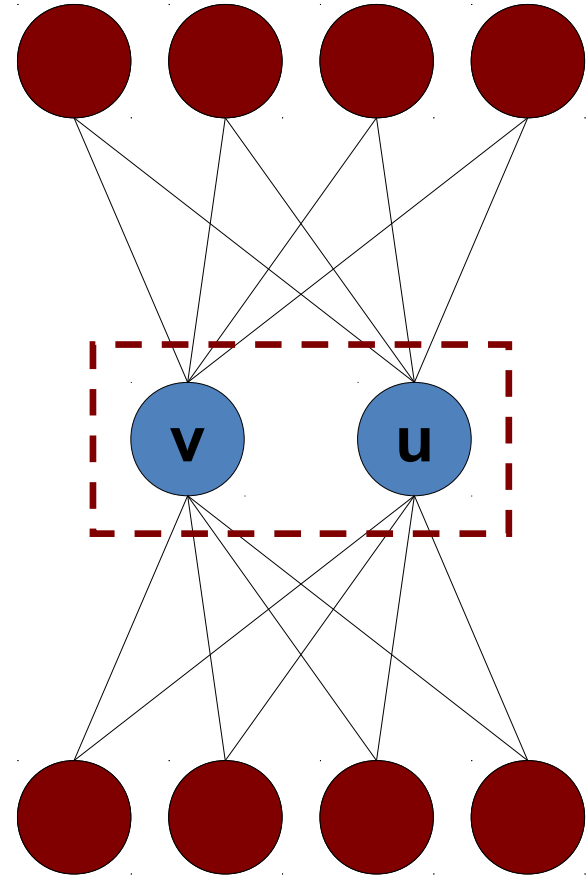
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- **Group degree centrality** – how many neighbours the group of nodes **v** and **u** have



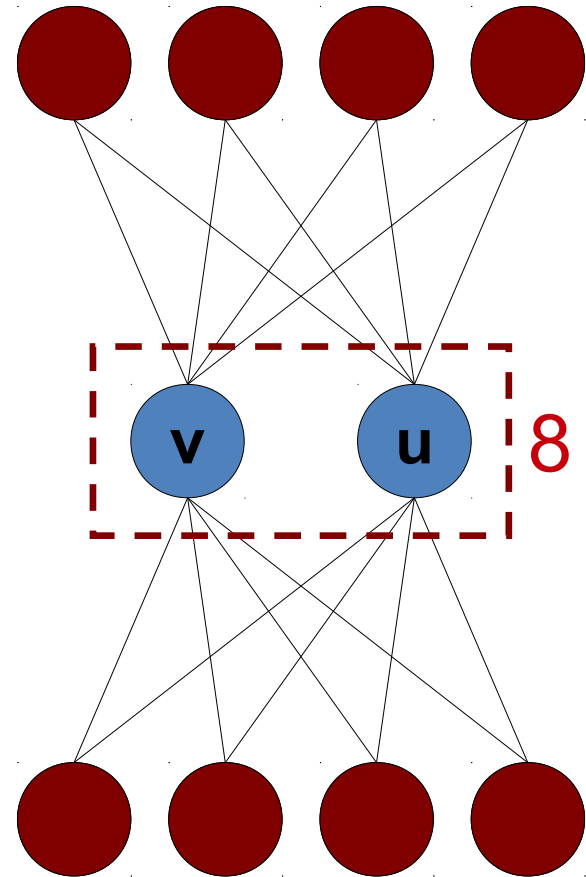
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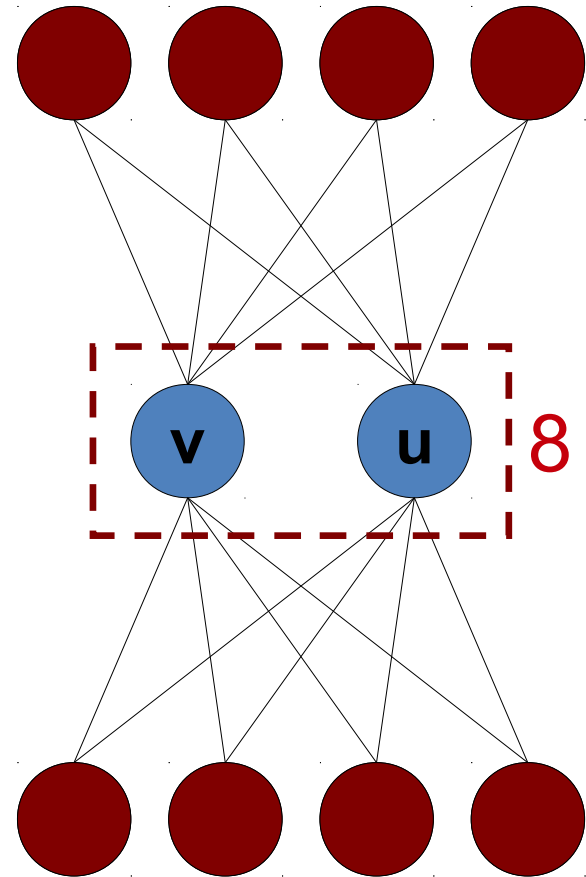


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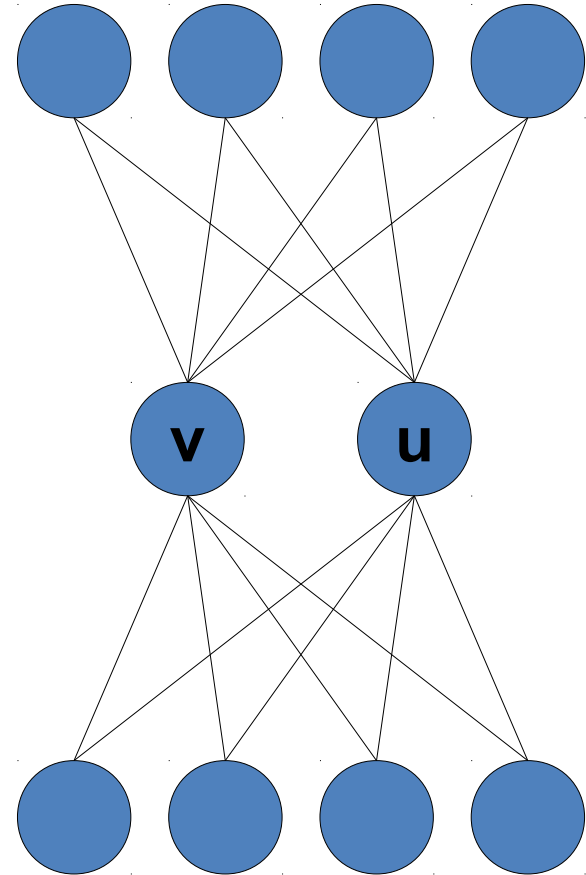
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Application:

- Find important people in Social Networks

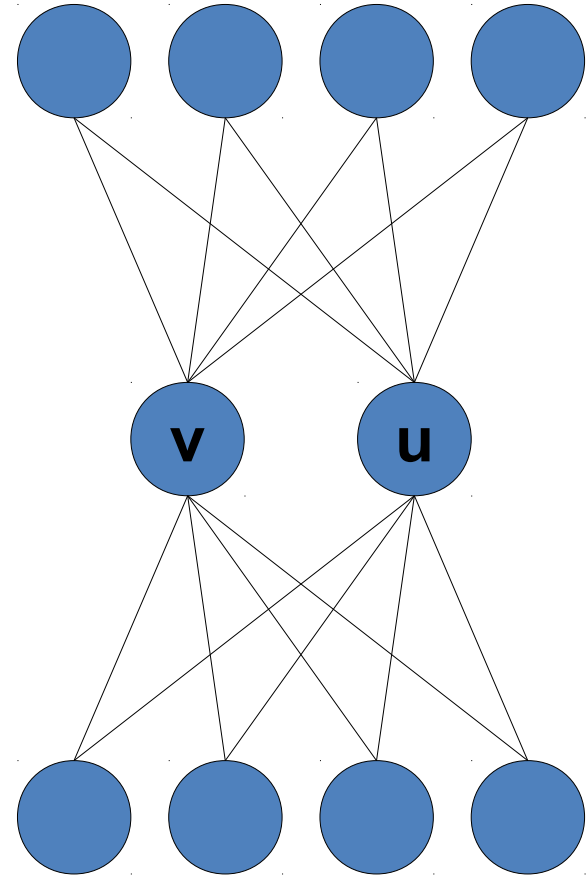


Closeness centrality



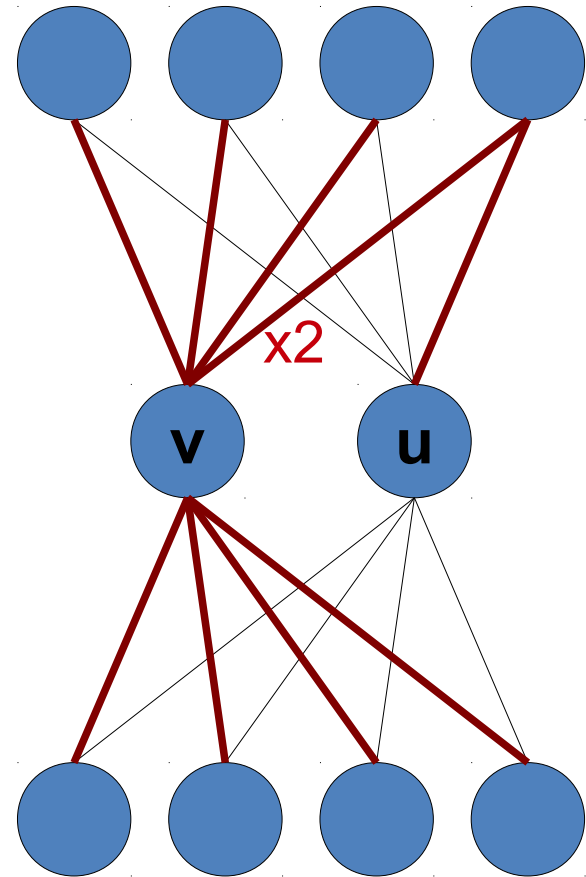
Closeness centrality

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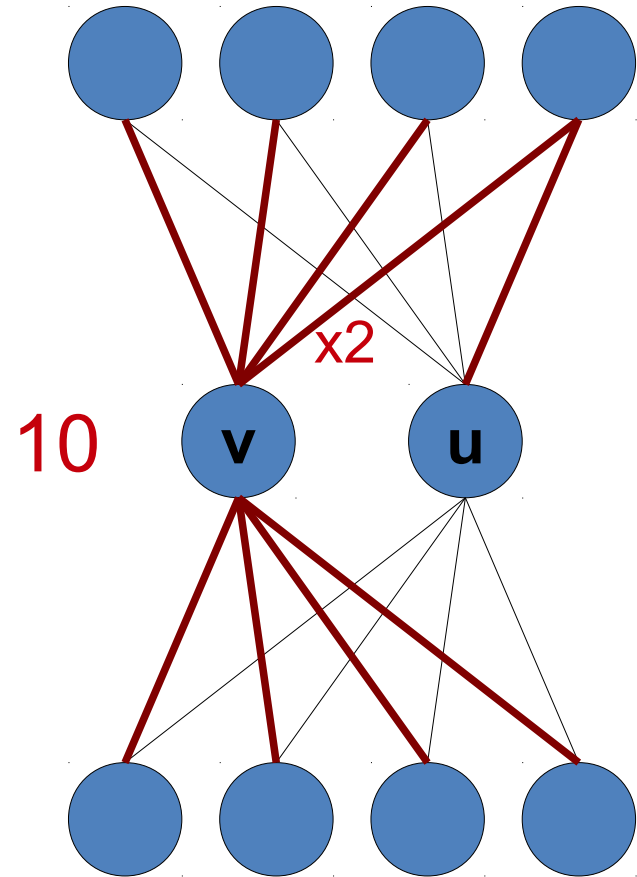
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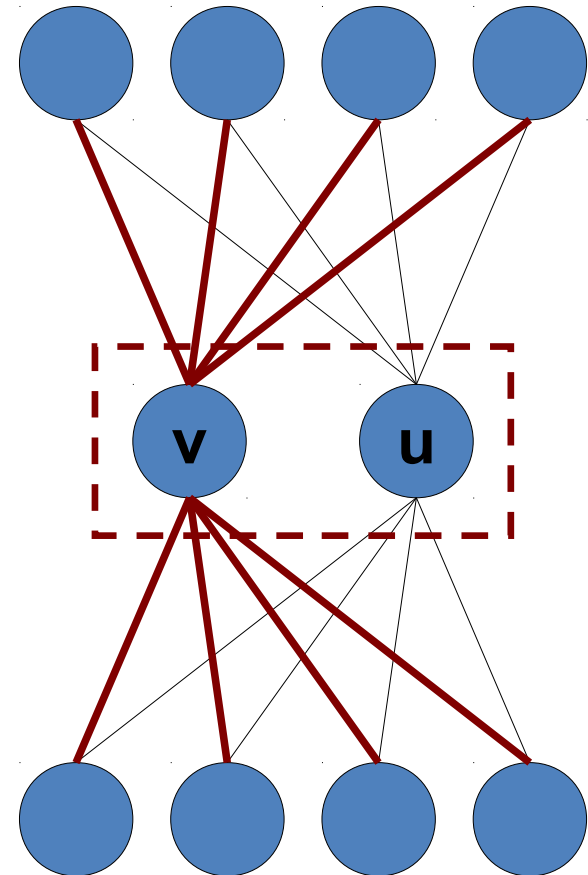
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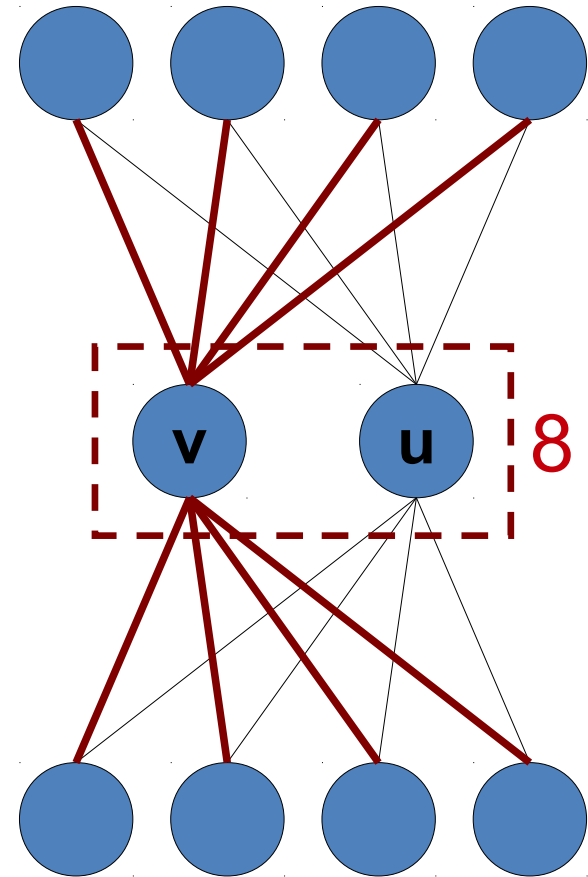
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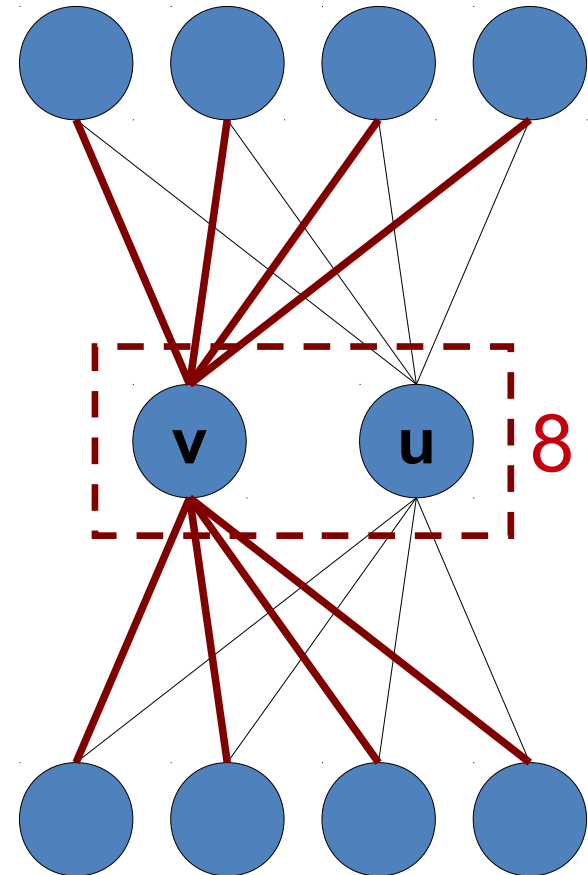


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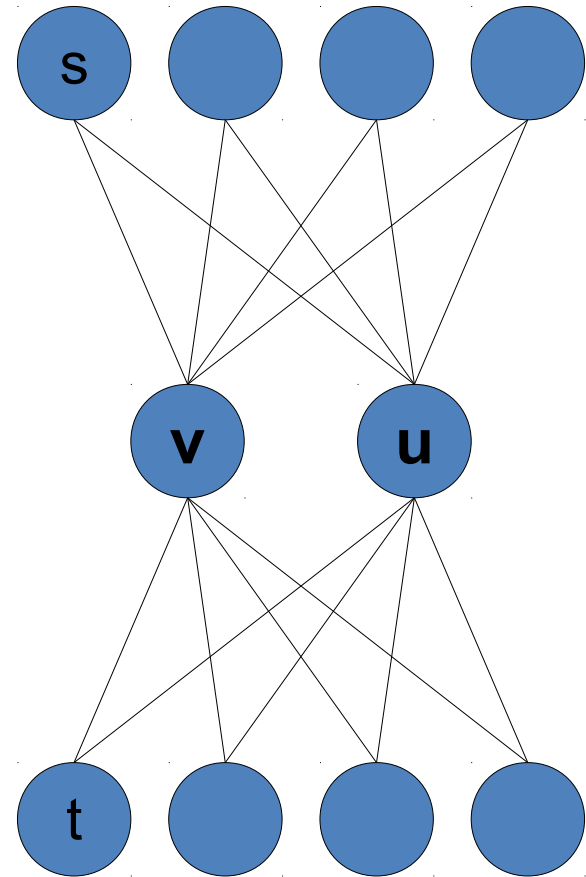
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Application:

- Find important node in the process of dissemination of information

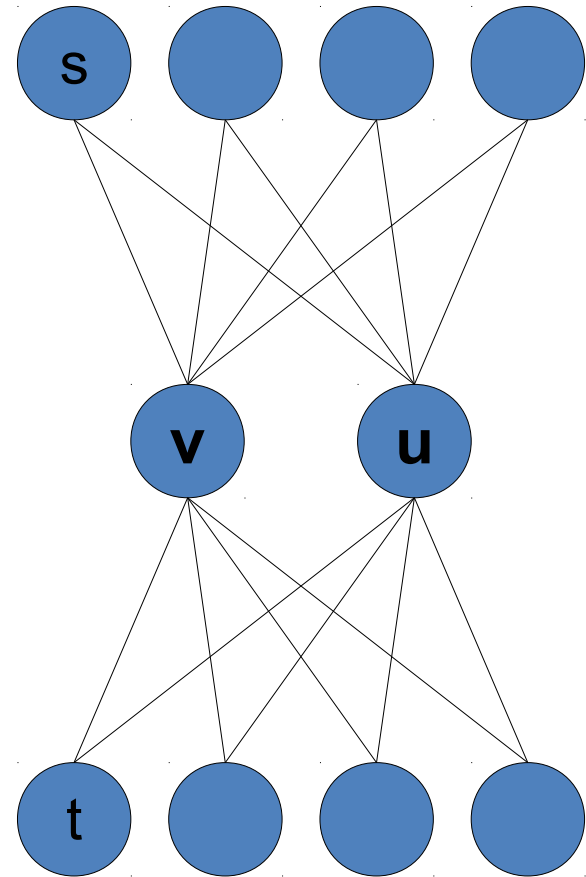


Betweenness centrality



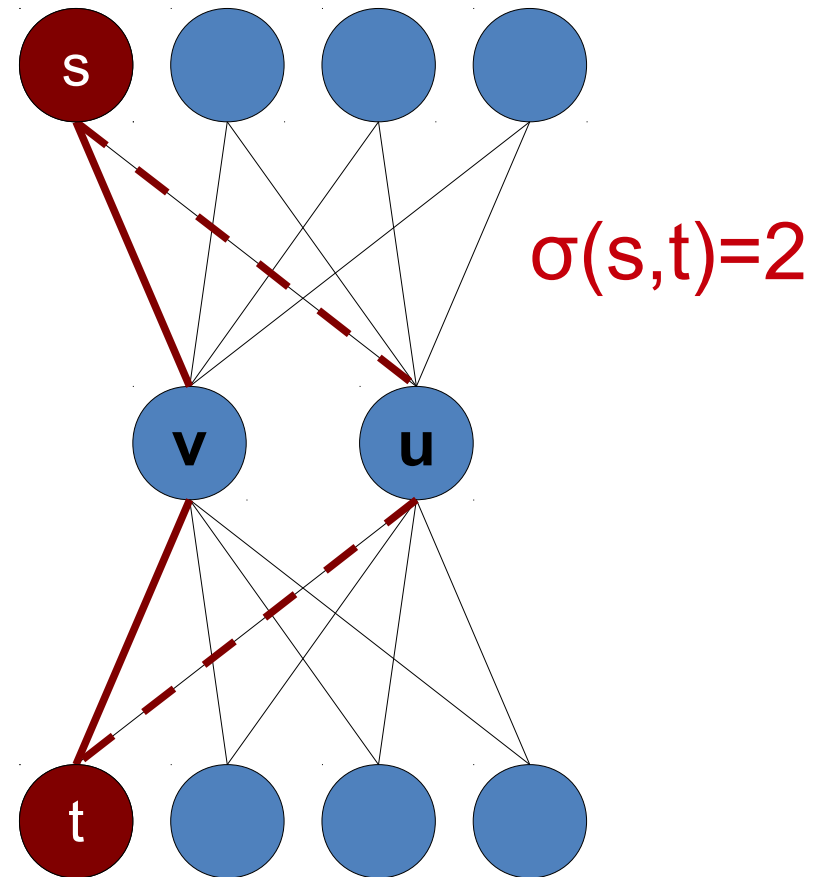
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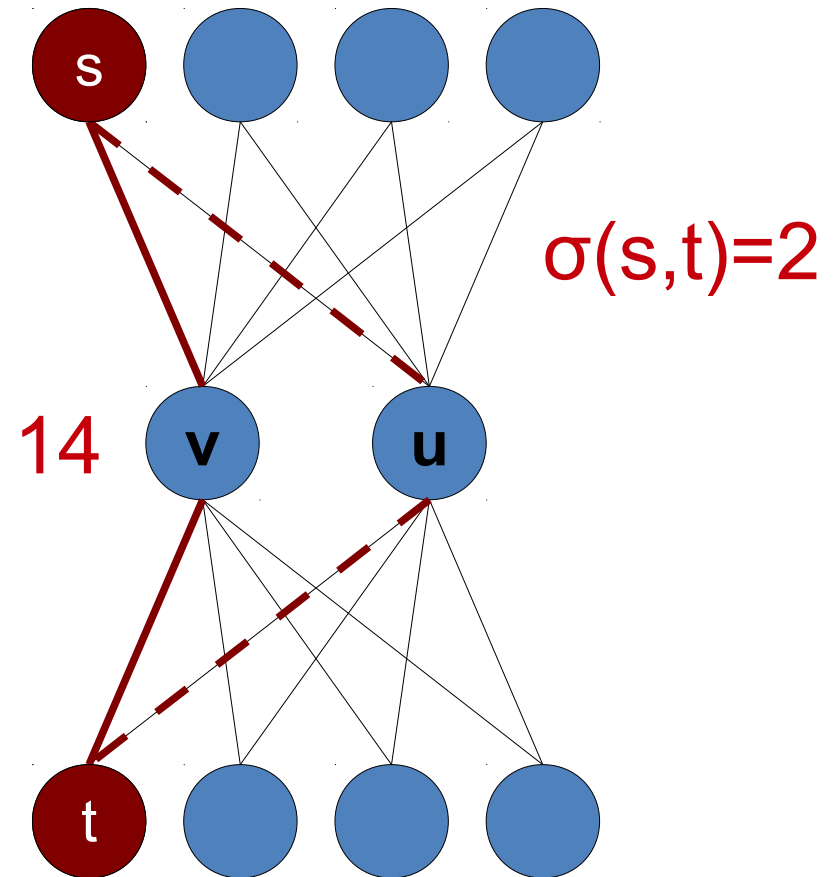
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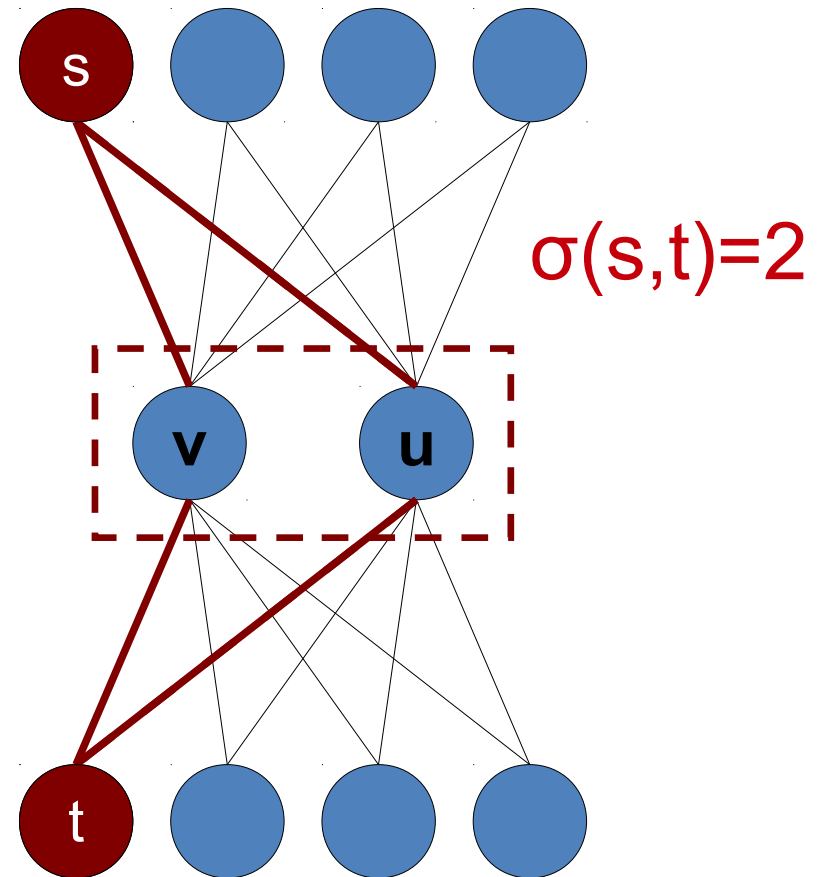
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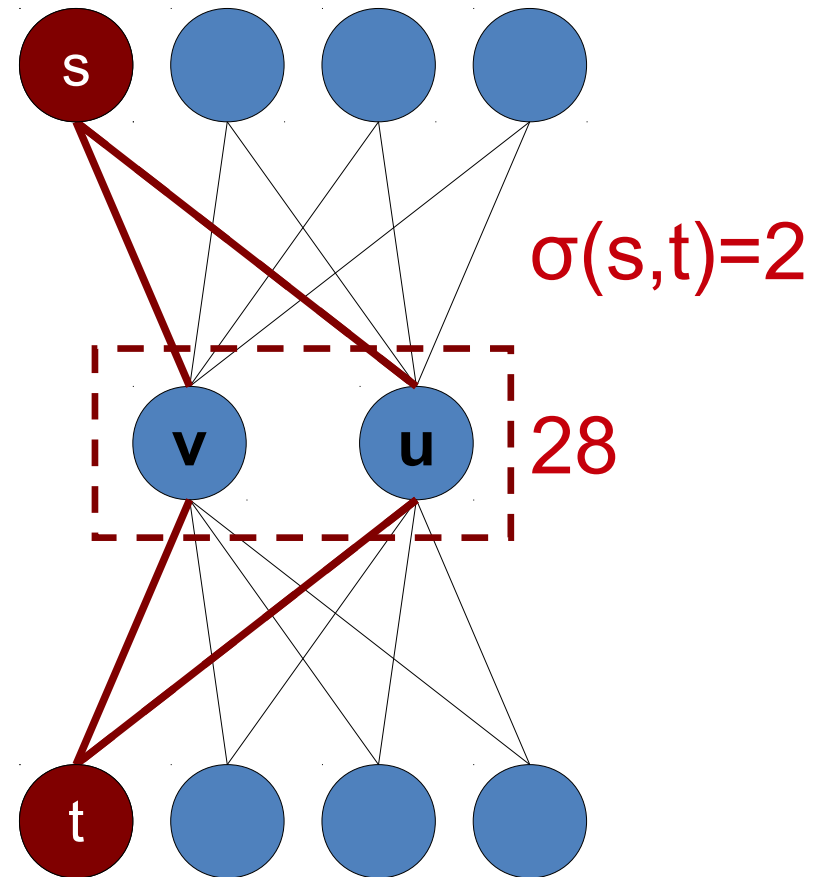
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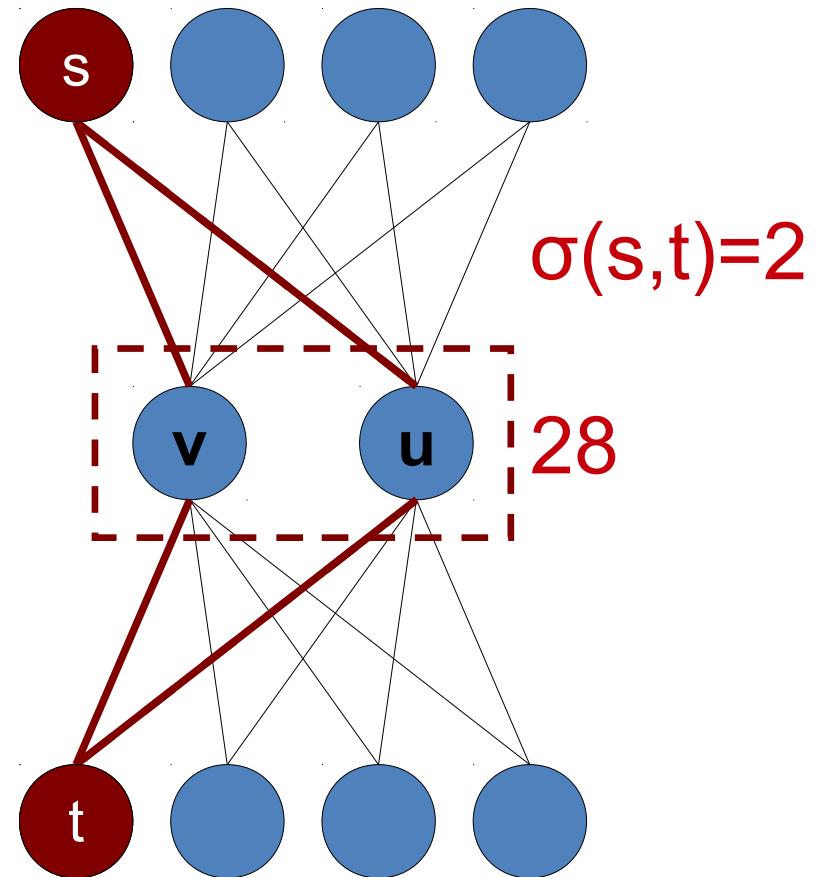


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Bird's-eye view

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Game Theory

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Shapley Value based centrality
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Applications and future work

Shapley Value

Shapley Value

Given 3 agents, the set of agents is:

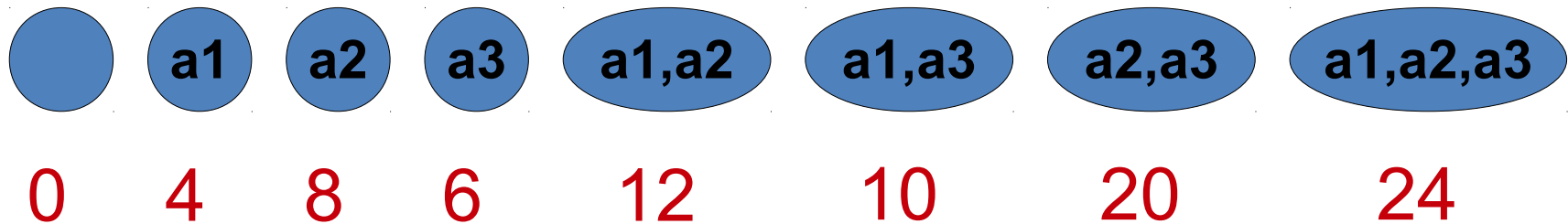
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Shapley Value

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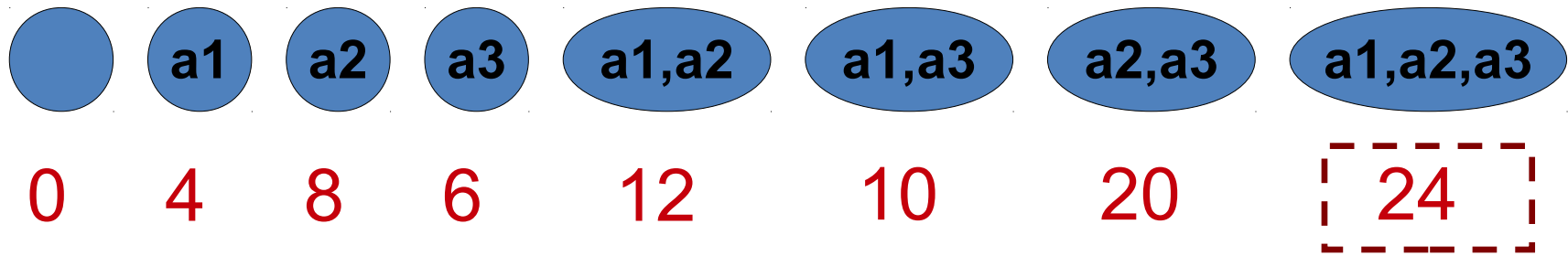


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How to distribute this payoff to players?

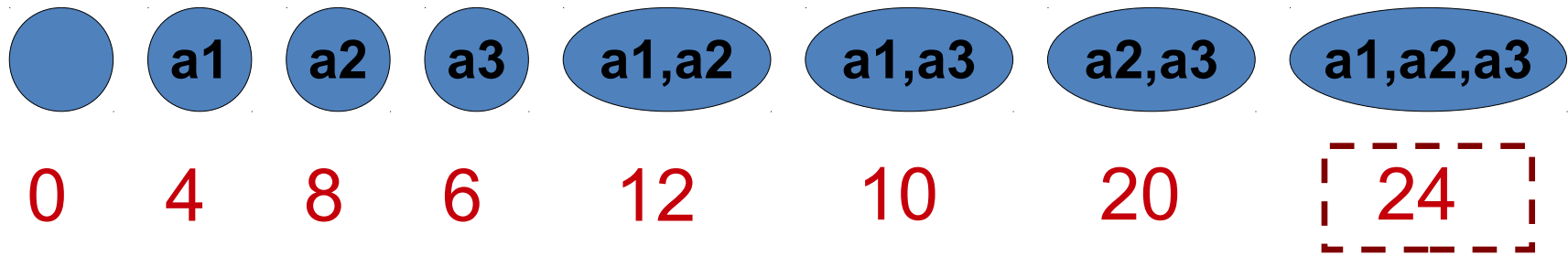
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→ **Shapley value** – a unique division of payoff that evaluates in a fair way a real contribution of each agent to the game

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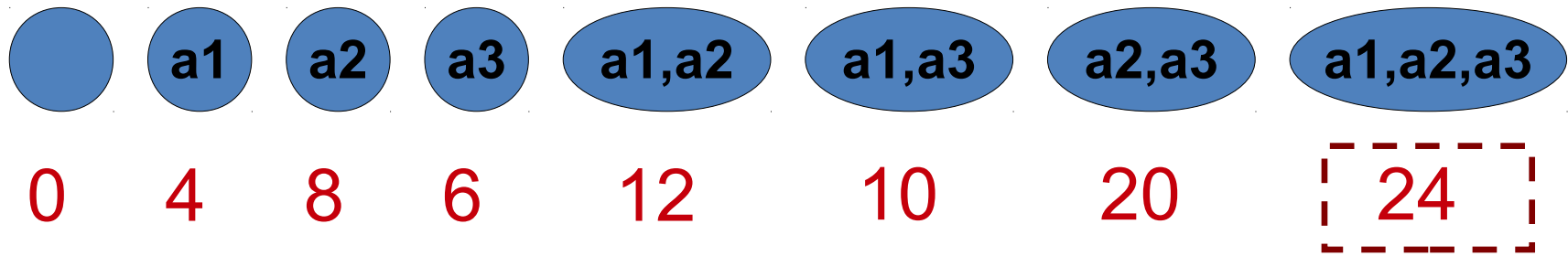
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Shapley Value - axioms

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→ **Symmetry** – nobody is in better position

$$\forall_{(S \subseteq N \setminus \{i, j\})} v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S) \Rightarrow \\ \Rightarrow SV_i(v) = SV_j(v)$$

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→ **Efficiency** – all surplus is distributed

$$\sum_{i \in N} SV_i(v) = v(N)$$

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→ **Linearity** – surplus from games can be sum up

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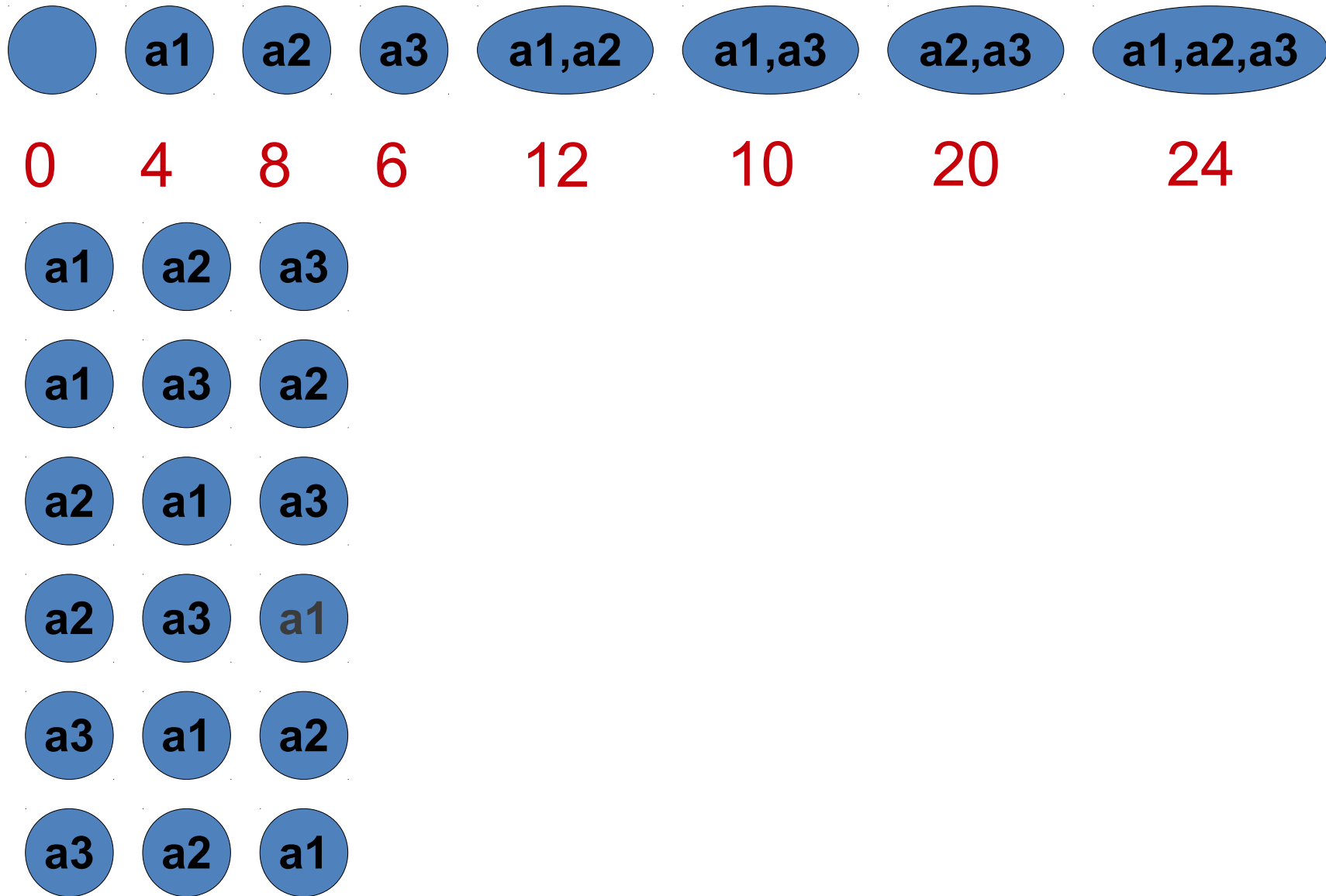
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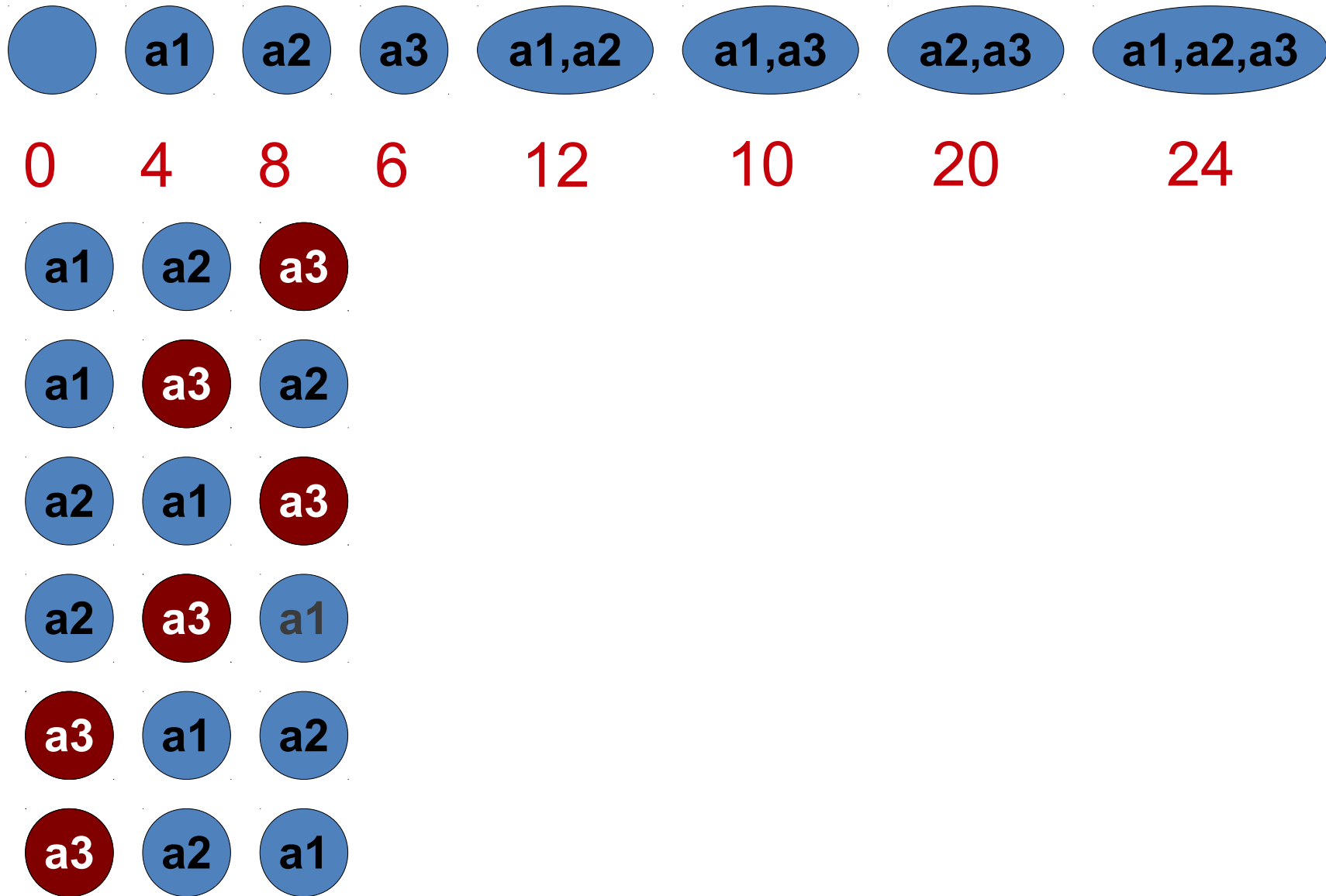
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→ **Shapley value** – a unique division of payoff that meets these three axioms

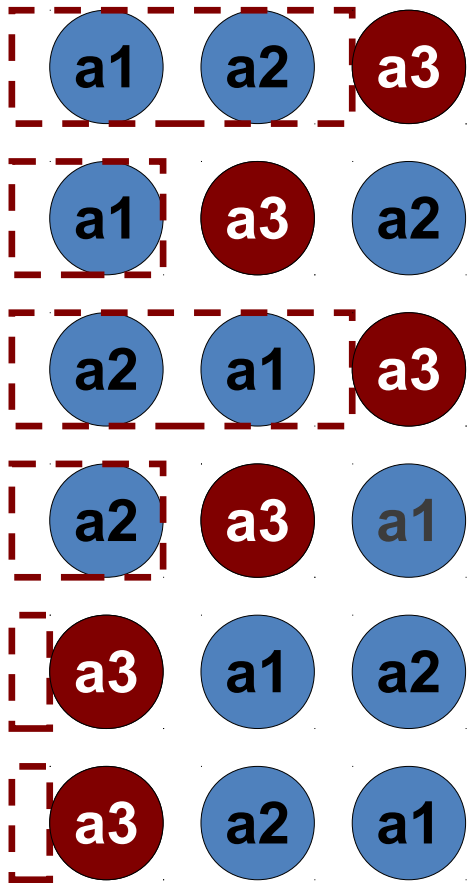
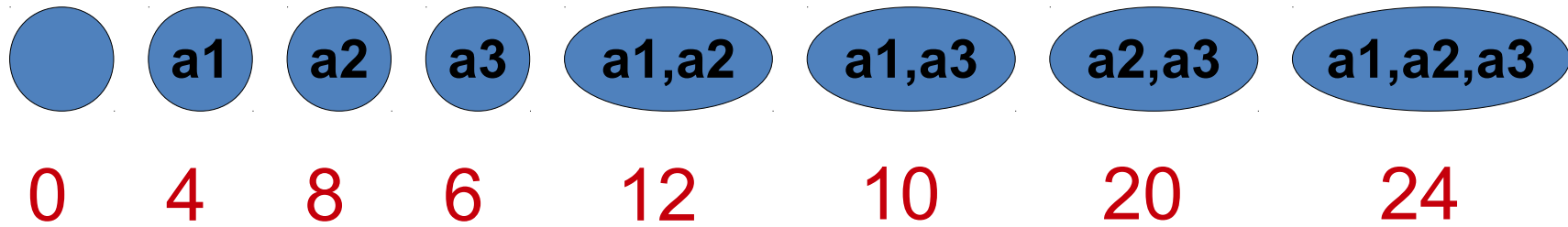
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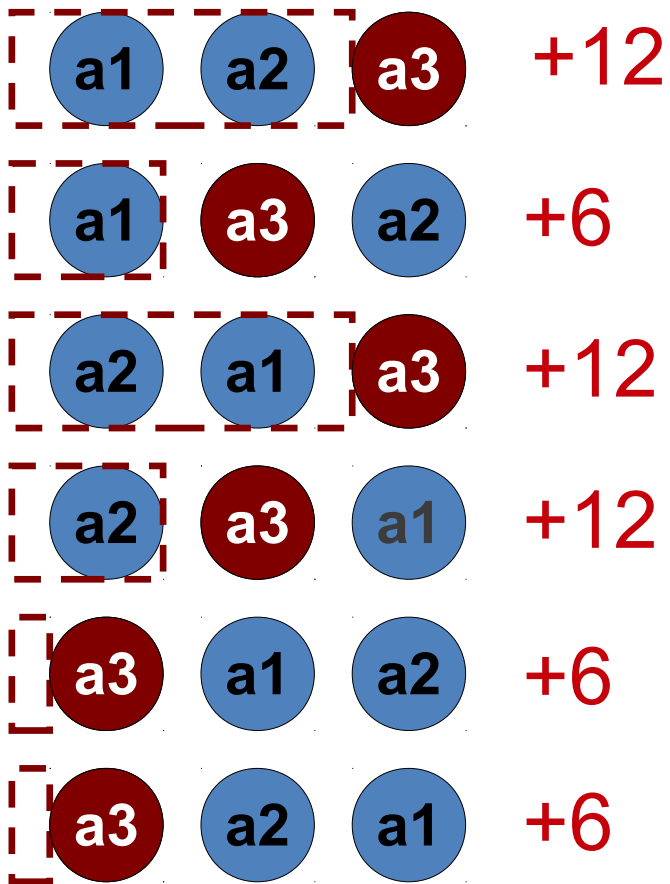
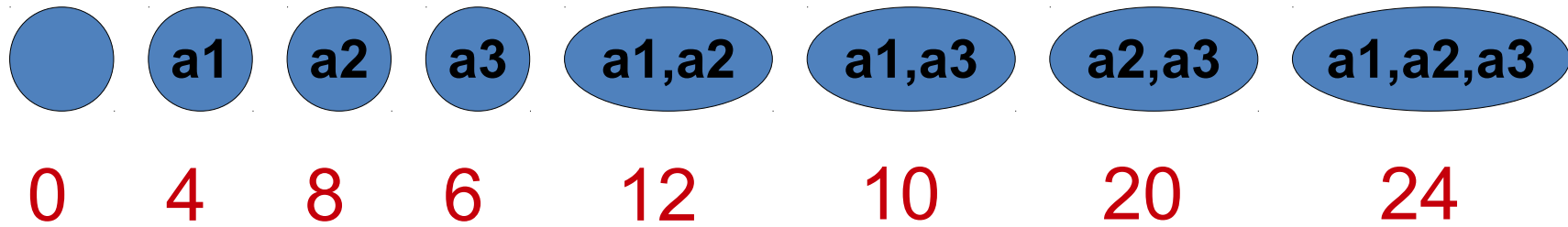
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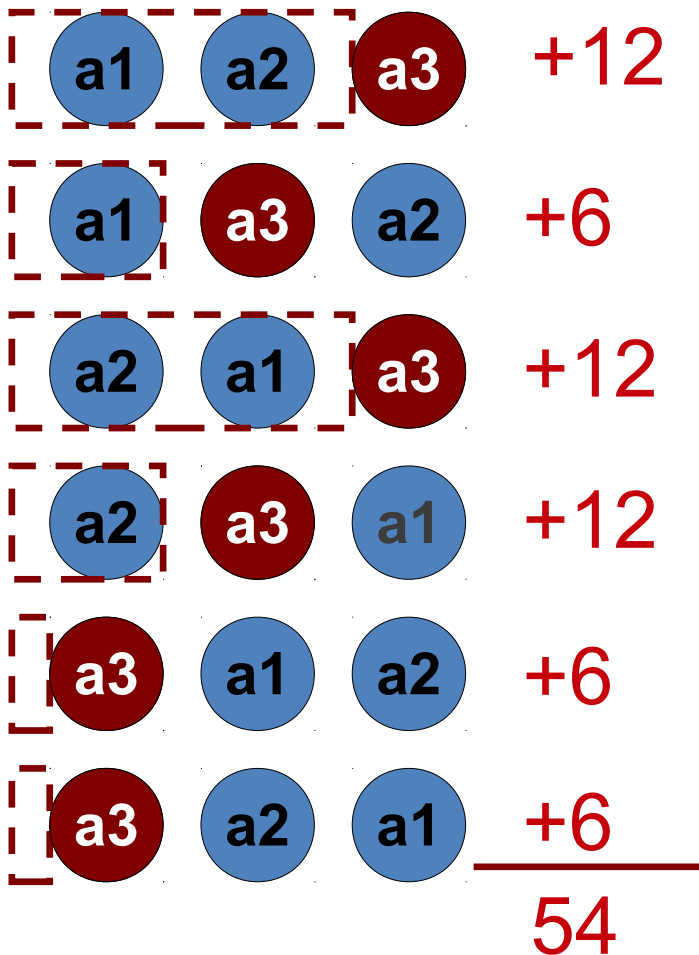
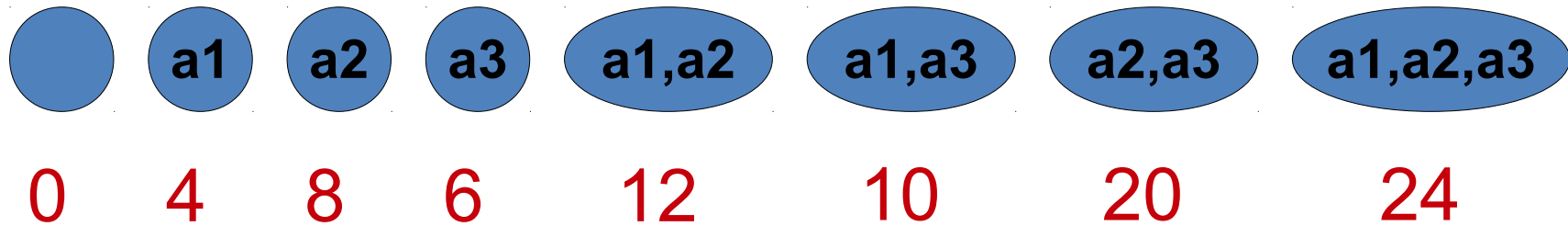
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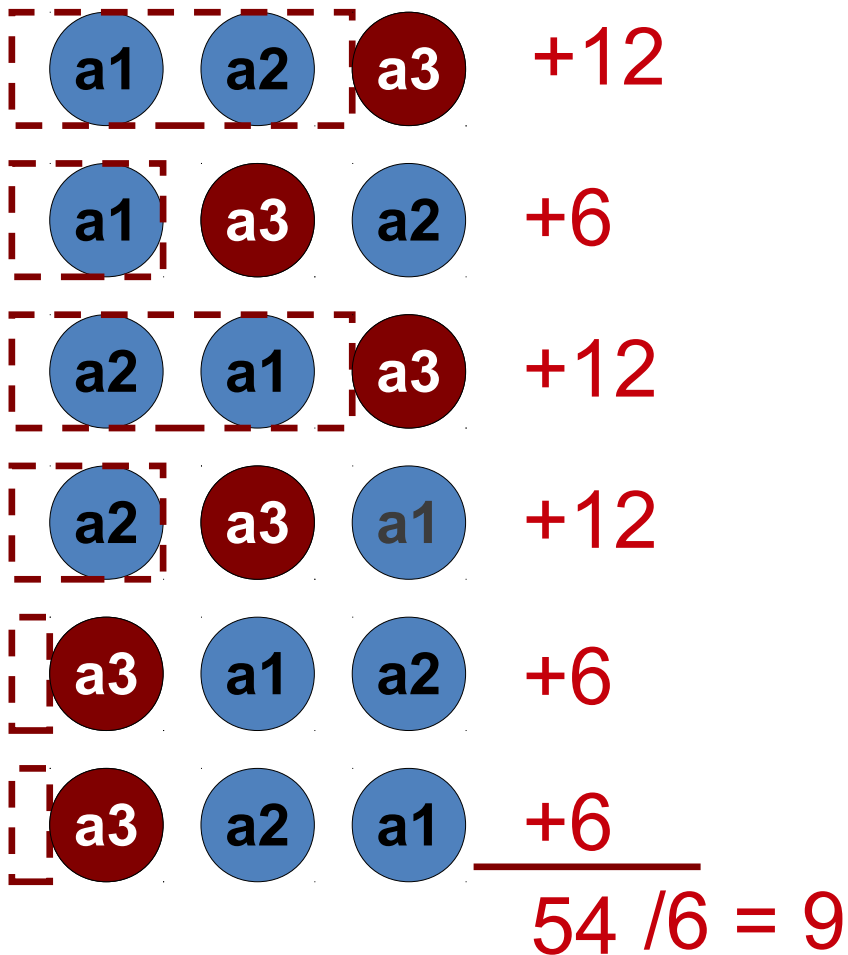
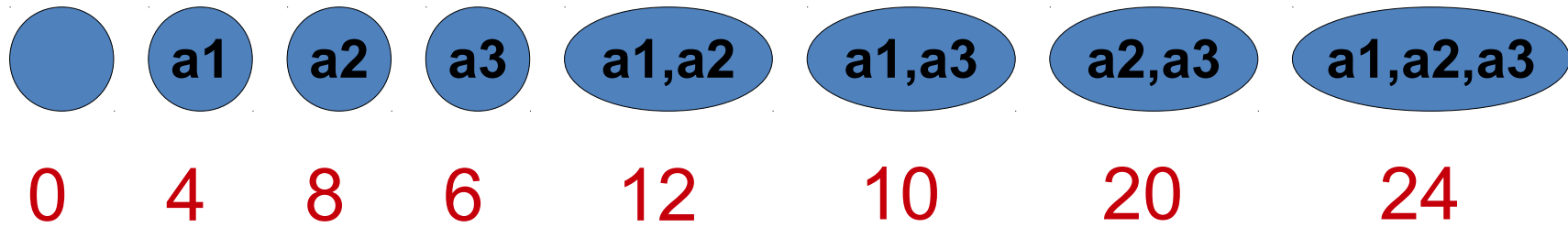
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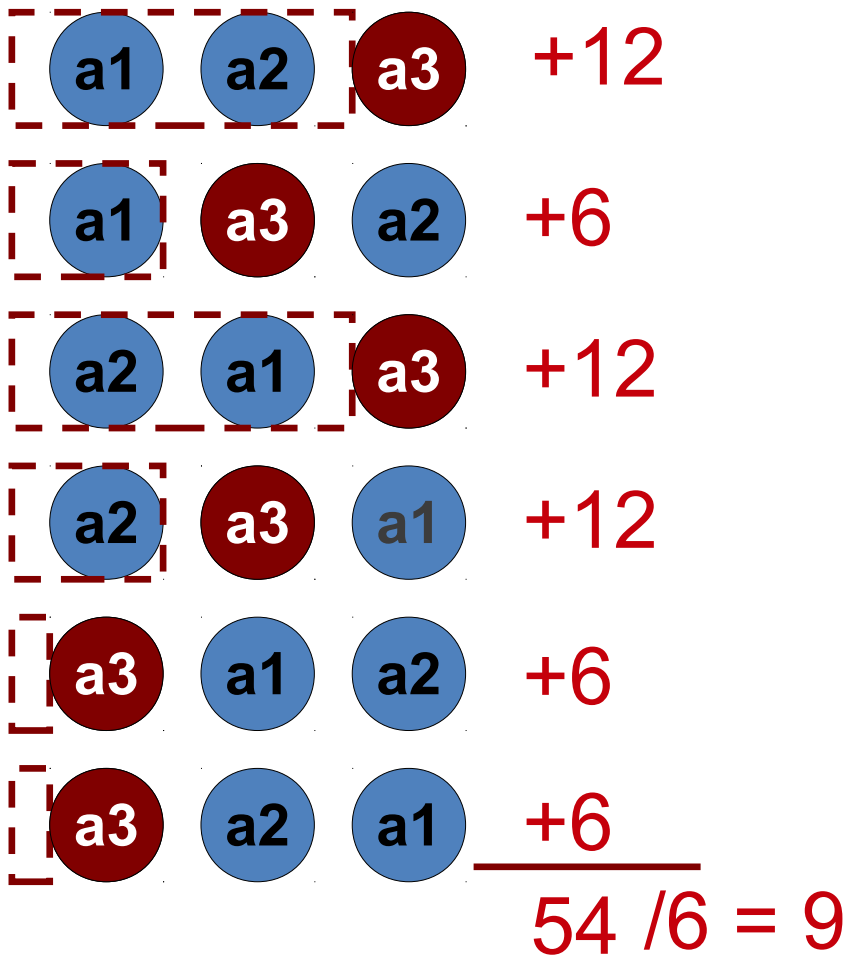
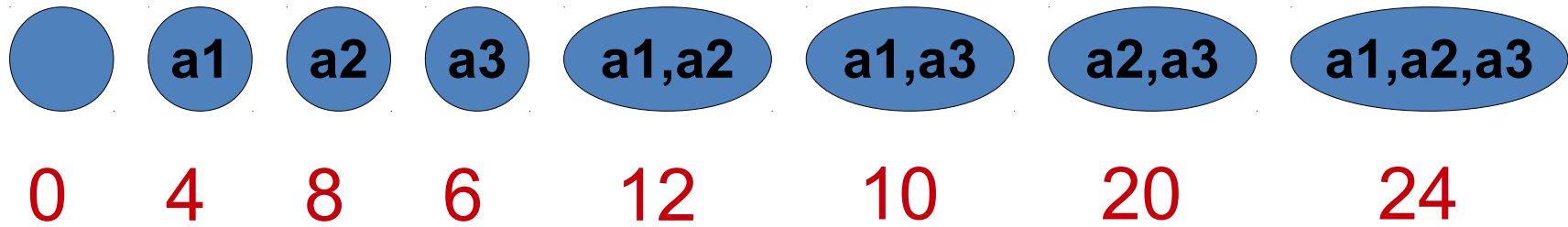
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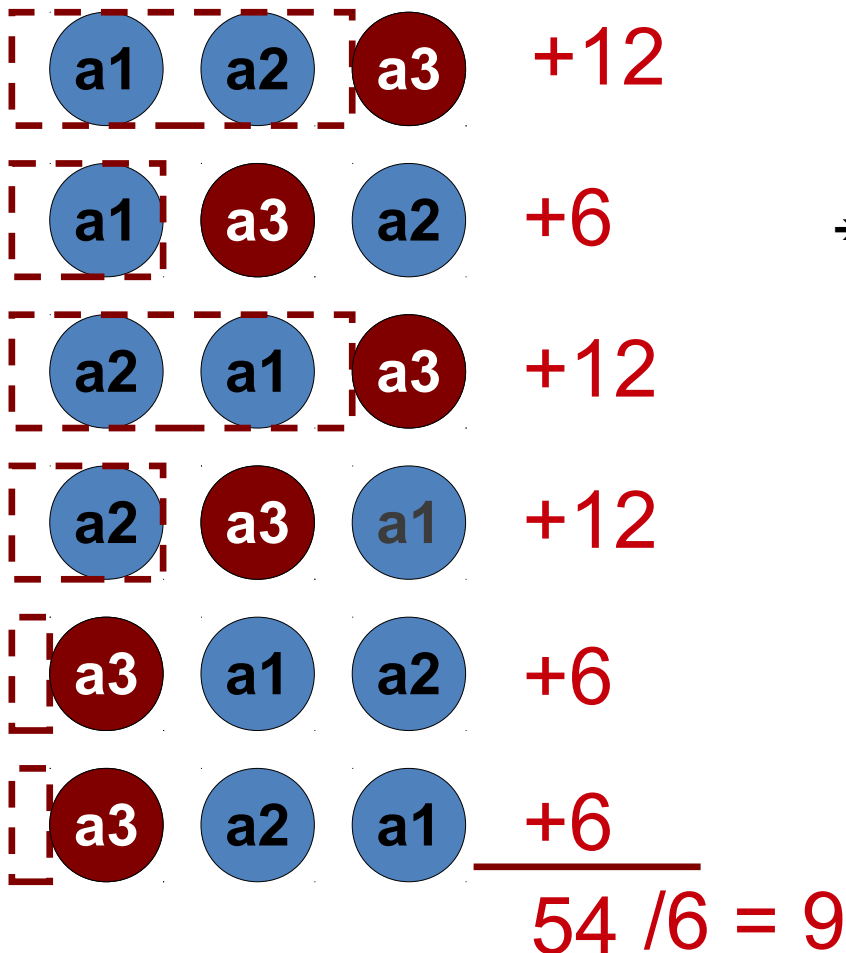
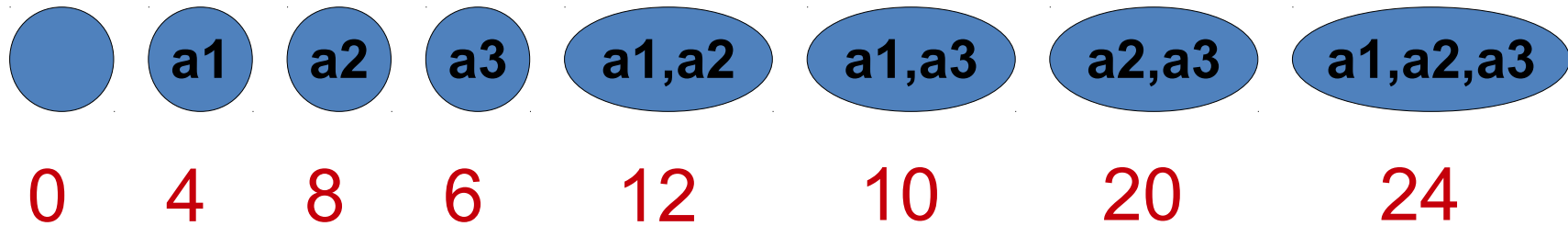


Shapley Value



$$SV_{a_3}(v, N) = 9$$

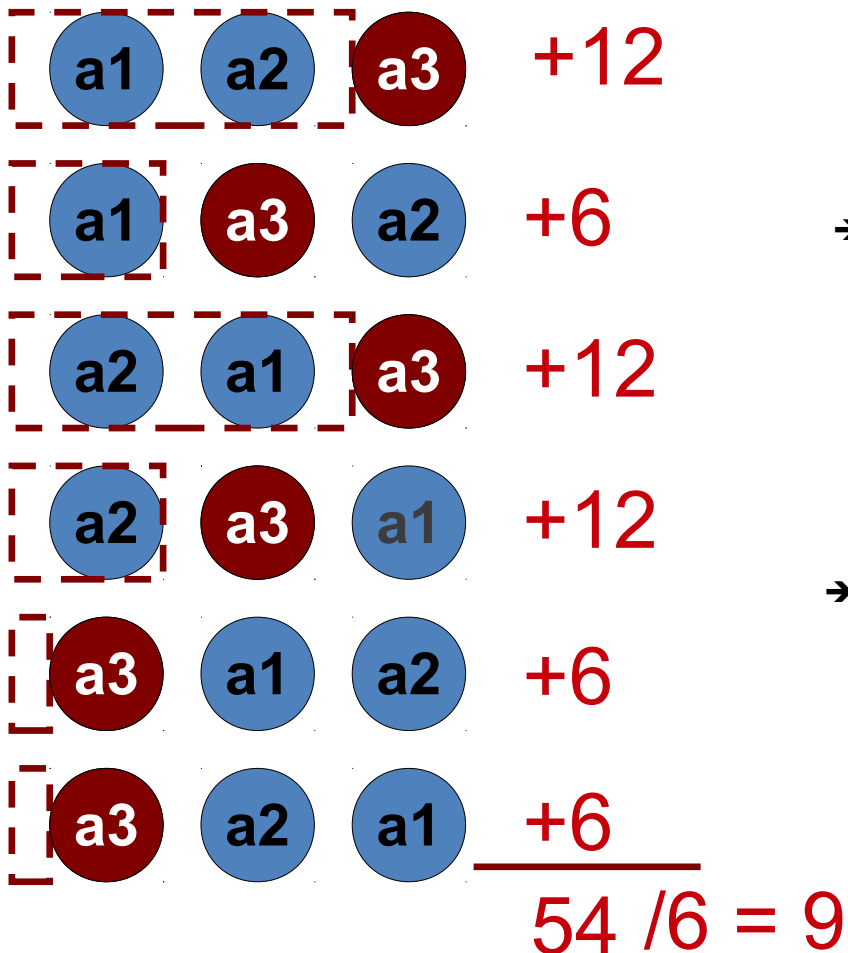
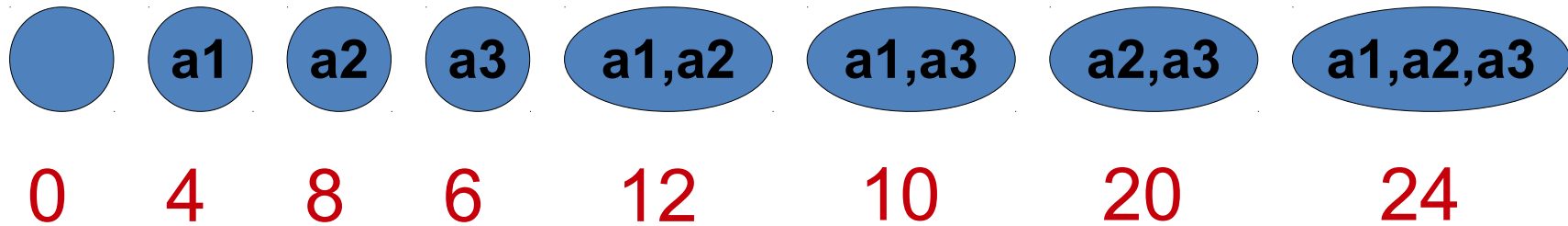
Shapley Value



$$SV_{a3}(v, N) = 9$$

→ **Shapley value of the node v** is an expected marginal contribution of v to the sets of nodes occurring before v in a random permutation

Shapley Value



$$SV_{a3}(v, N) = 9$$

- **Shapley value of the node v** is an expected marginal contribution of v to the sets of nodes occurring before v in a random permutation
- **Computational challenge** – in general case we need the exponential number of calculations

$$O(2^{|N|})$$

Bird's-eye view

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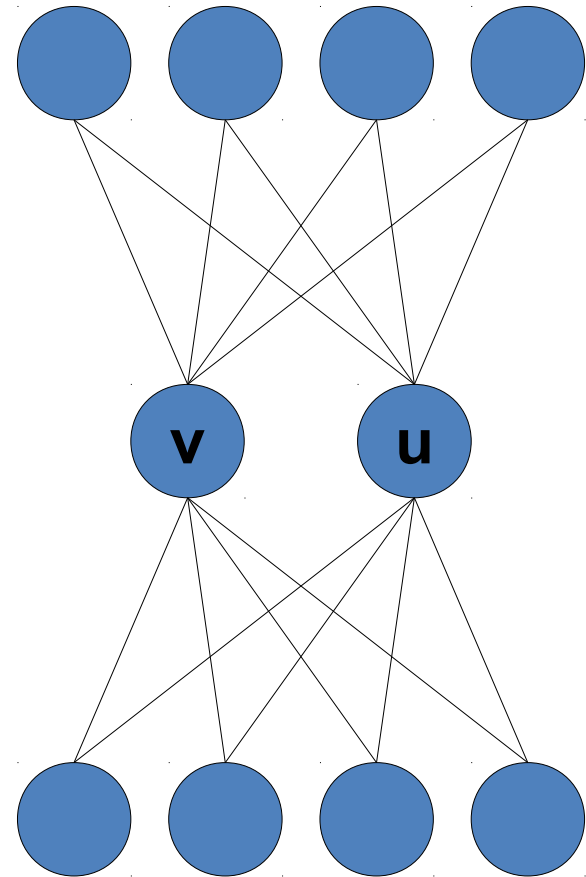
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Shapley Value based centrality
metrics

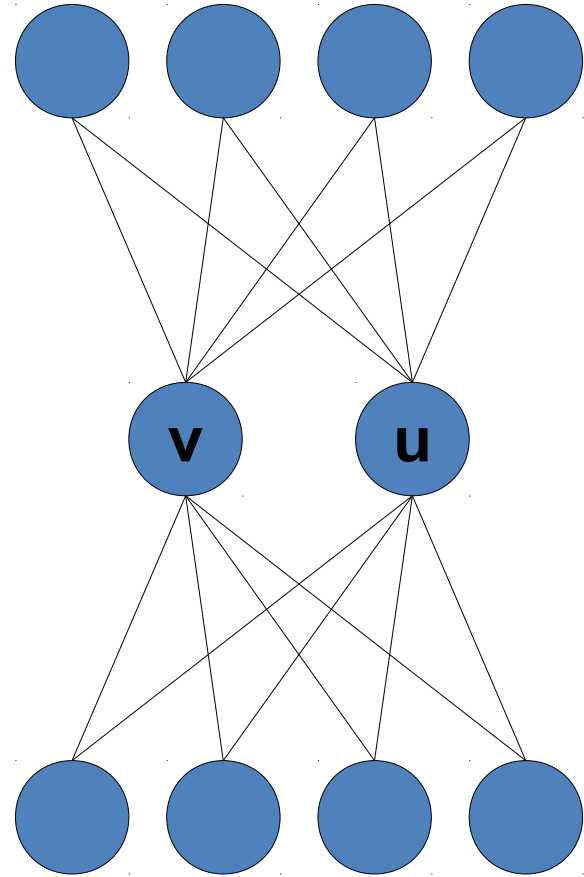
Applications and future work

Shapley Value based centrality



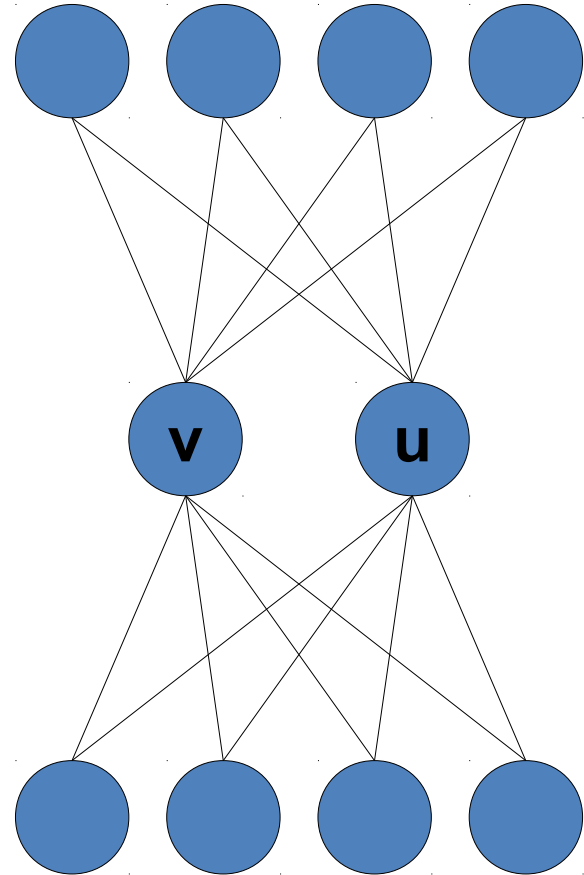
Shapley Value based centrality

→ **Players in the game** – nodes in the graph



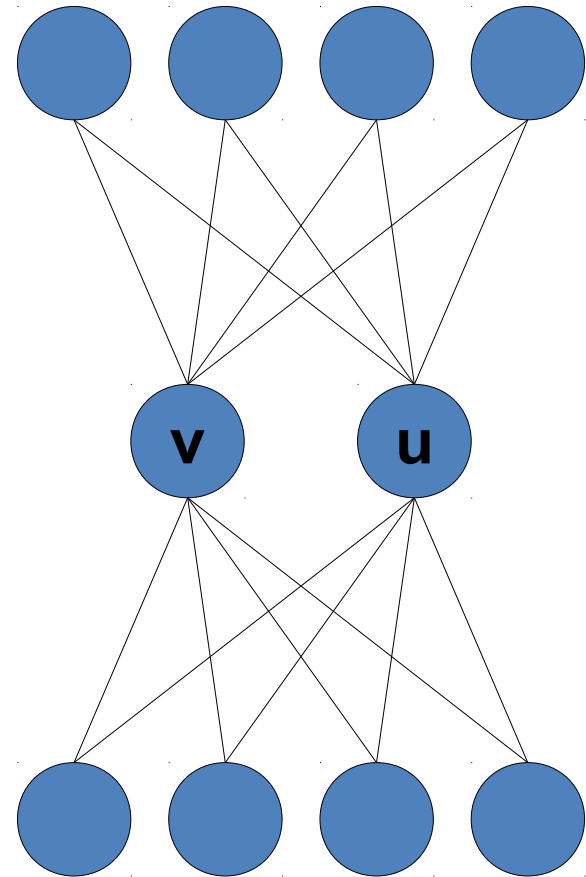
Shapley Value based centrality

- **Players in the game** – nodes in the graph
- **Characteristic function** – standard centrality but for coalition of nodes (group centrality)



Shapley Value based centrality

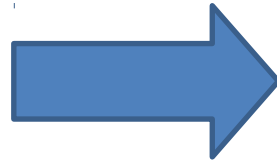
- **Players in the game** – nodes in the graph
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- **Shapley value gives value of individual node** – evaluation takes into account all other coalitions in the graph



Centralities based on the Shapley value

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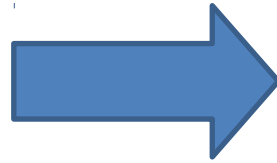
General case



$$O(2^{|V|})$$

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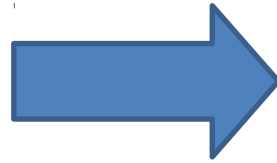
Group degree
centrality



$$O(|V| + |E|)$$

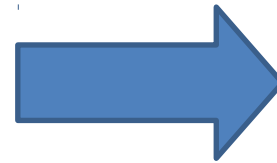
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Group degree
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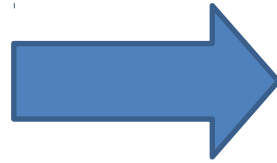
Group closeness
centrality



$$O(|V||E|)$$

Centralities based on the Shapley value

General case



$$O(2^{|V|})$$

Group degree
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$$O(|V|+|E|)$$

Group closeness
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$$O(|V||E|)$$

Group betweenness
centrality

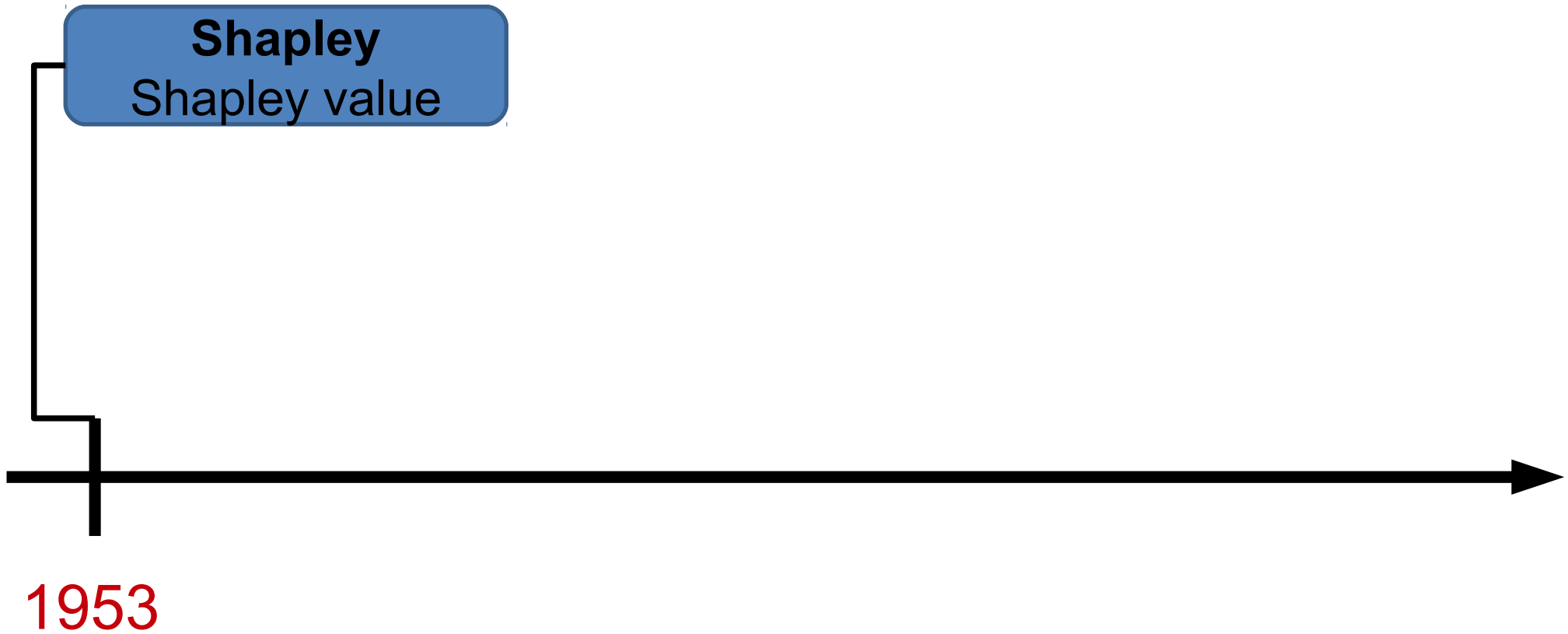


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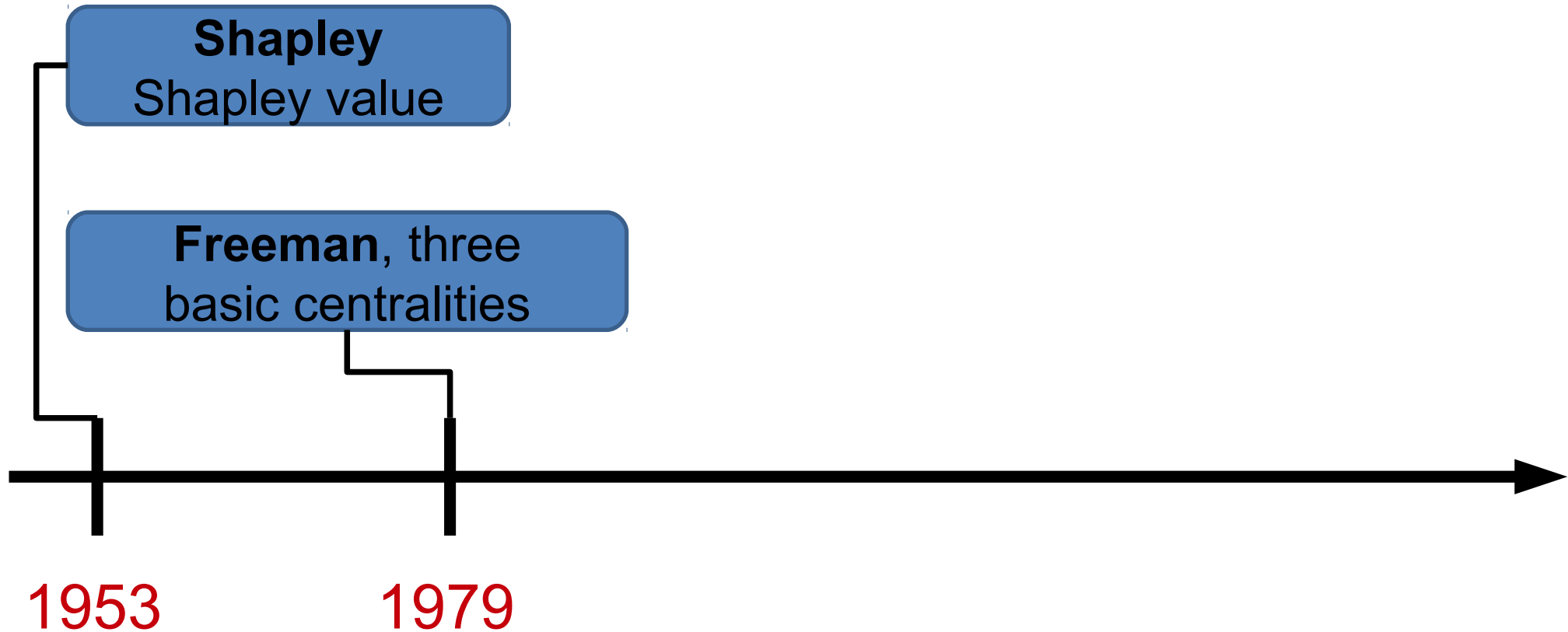
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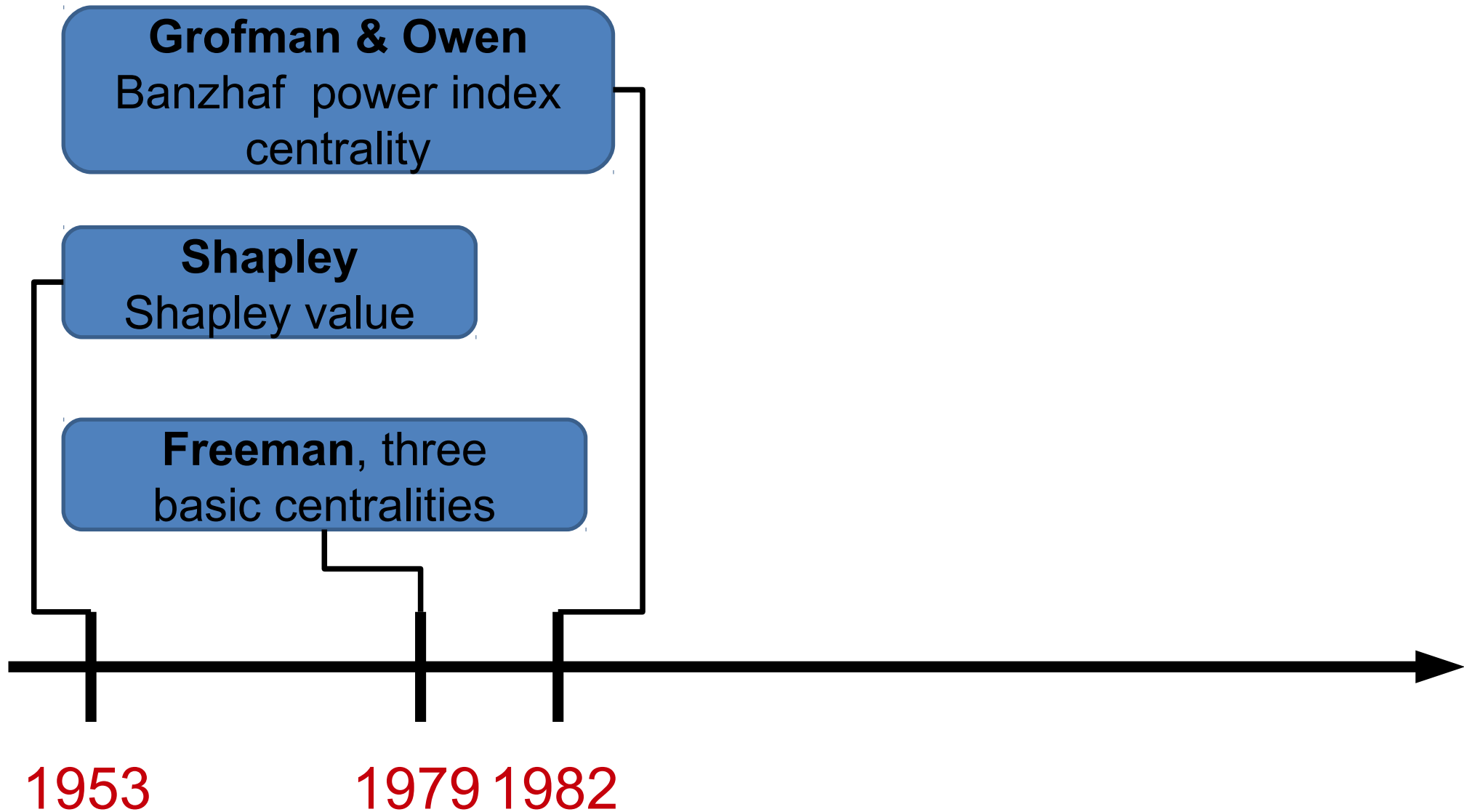
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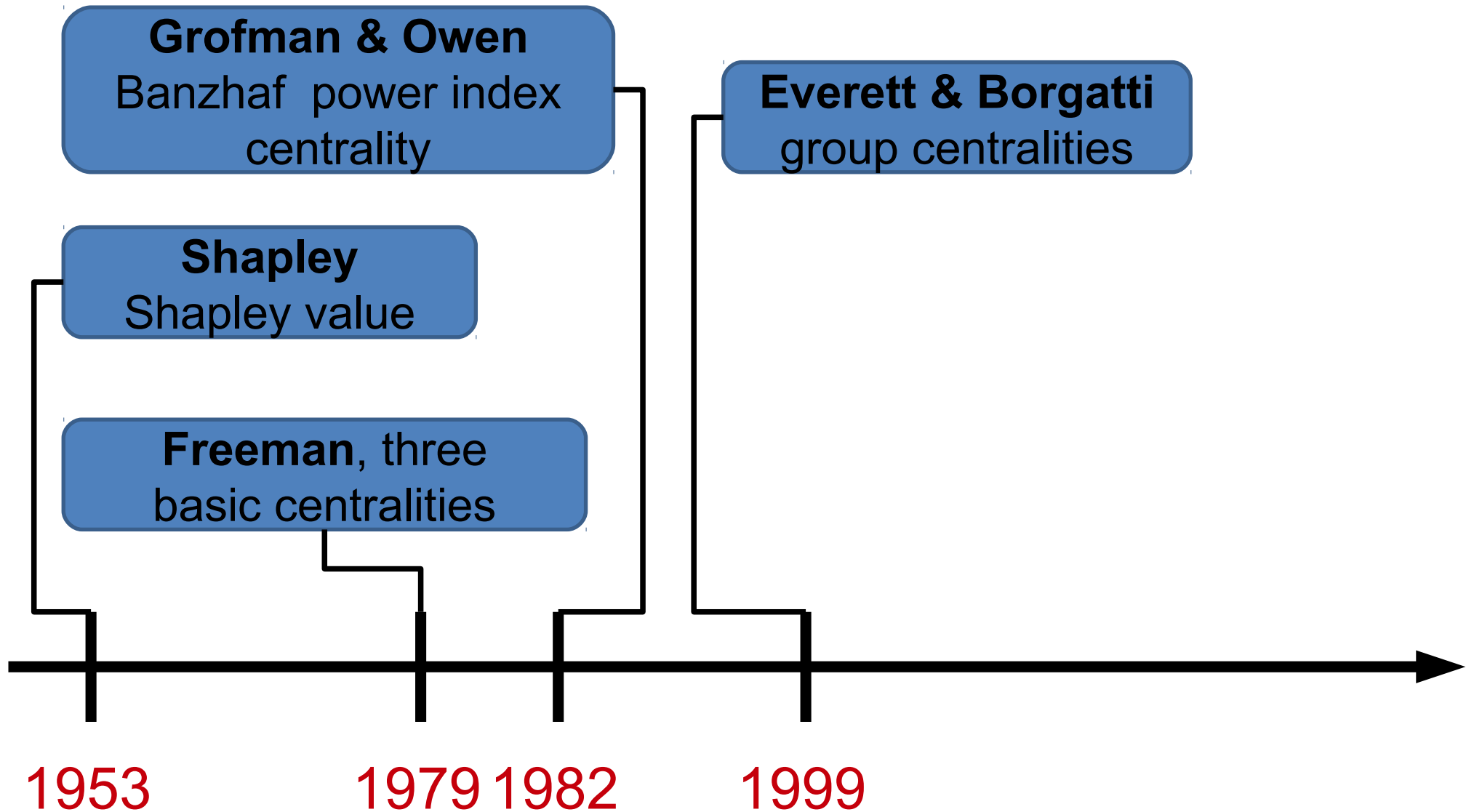
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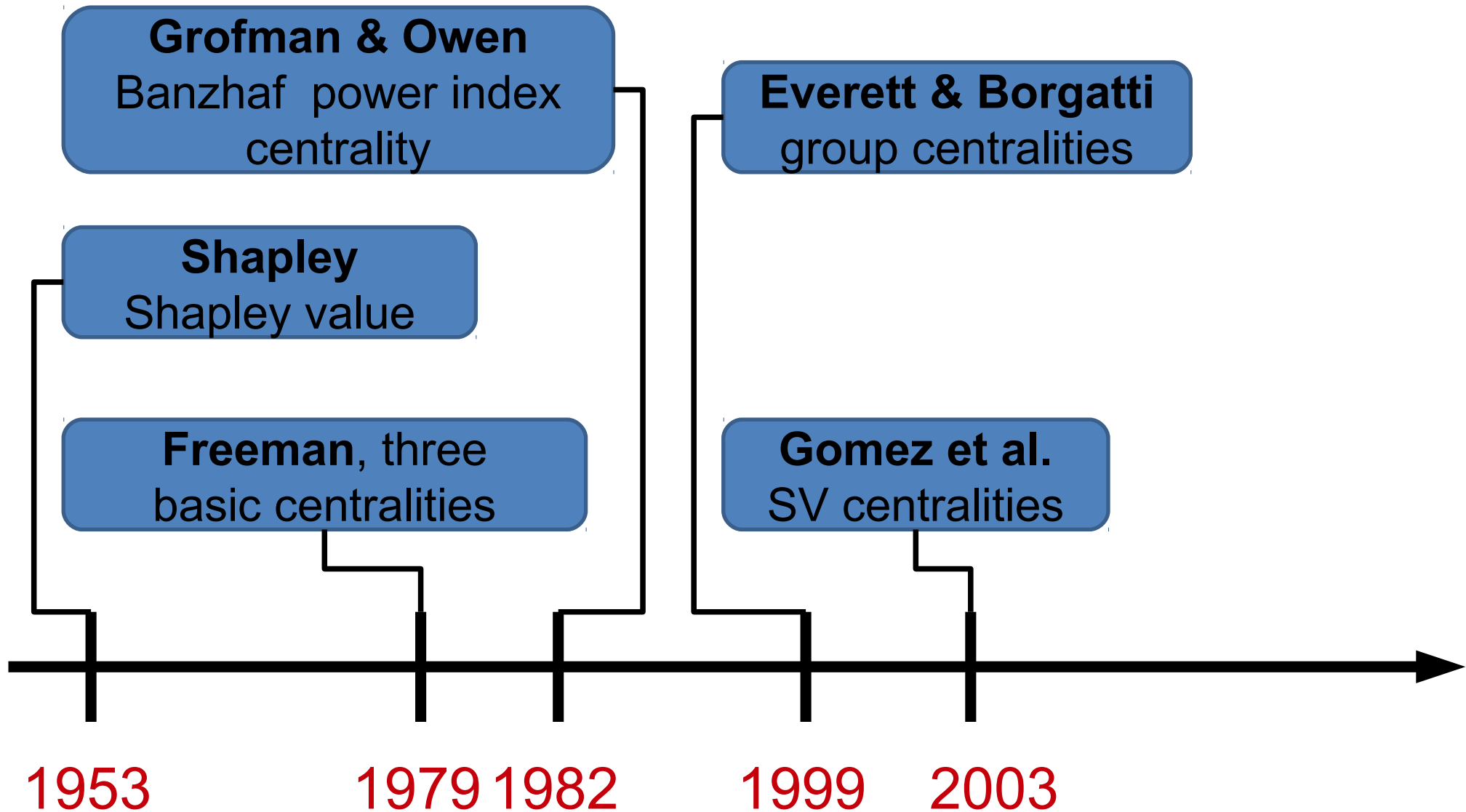
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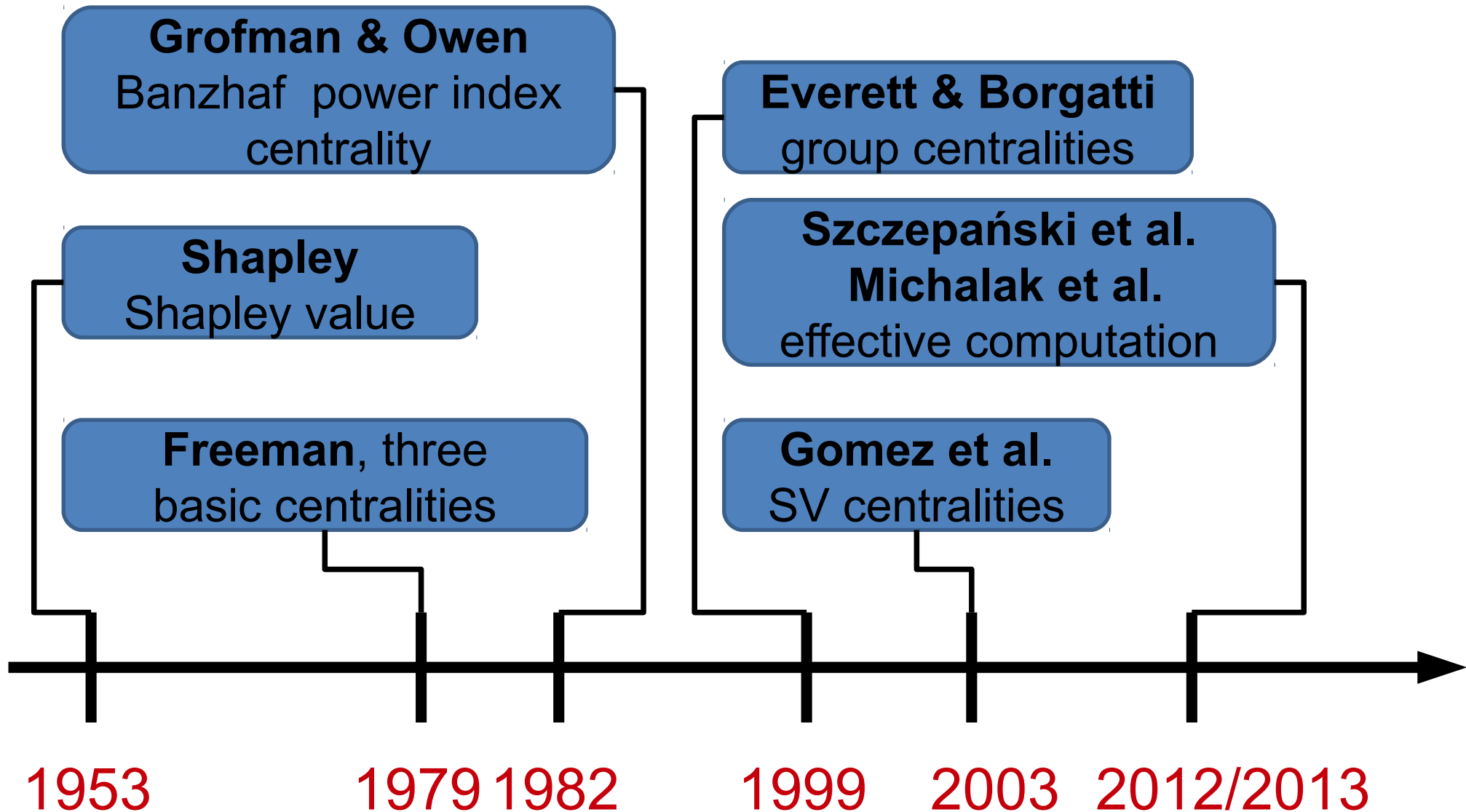
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History



History



Bird's-eye view

Graph Theory

Game Theory

Problem:

how important is a node in a network?

Problem:

how important is a player in a game?

Answer: Centrality
Metrics

Answer: Shapley Value

Shapley Value based centrality
metrics

Applications and future work

k-nodes problem

How to choose the best k-nodes in order to maximize information propagation?

N.R. Suri, & Y. Narahari, 2010

Viral
marketing

Graph-theoretic
degree centrality

Terrorist Network Analysis

How to choose most influential people in terrorist networks?

Lindelauf et al., 2013

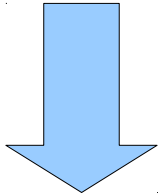
Support for
security
agencies

Shapley value and
connectivity games

What next?

What next?

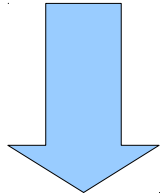
Generalized
coalitional games



Cascading models
of centralities

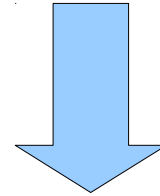
What next?

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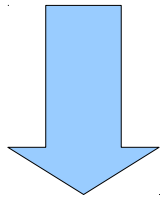
Game-theoretical centralities



Community
detection

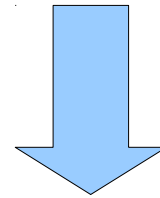
What next?

Generalized
coalitional games

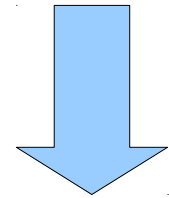


Cascading models
of centralities

Game-theoretical centralities



Community
detection



Inoculation
strategies

Thank You for Your attention!